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then x and b are apolar to ζ and $1/\zeta$; and the line bx, being the polar of a as to R, has the equation

$$\Sigma\, ay/\sqrt{ax} = 0,$$

and is the tangent of F at x.

Dually then, ζ and $1/\zeta$ being the absolute points, the conic F the Feuerbach circle, and the conic R a rectangular hyperbola on the four given orthocentric points, and having its centre c on F, if the common diameter of F and R meet R at points dd', then these points are double foci of circular curves of class 3 on the 6 lines; the circles with centres d and d' and touching F at c are the tritangent conics; and the two cubics touch F at c.

DEFORMATIONS OF TRANSFORMATIONS OF RIBAUCOUR

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When a system of spheres involves two parameters, their envelope consists in general of two sheets, say Σ and Σ_1, and the centers of the spheres lie upon a surface S. A correspondence between Σ and Σ_1 is established by making correspond the points of contact of the same sphere. In general the lines of curvature on Σ and Σ_1 do not correspond. When they do, we say that Σ_1 is in the relation of a transformation of Ribaucour with Σ, and vice-versa. For the sake of brevity we call it a transformation R.

It is a known property of envelopes of spheres that if the surface of centers S be deformed and the spheres be carried along in the deformation, the points of contact of the spheres with their envelope in the new position are the same as before deformation.[1] Ordinarily when S for a transformation R is deformed, the new surfaces Σ' and Σ'_1 are not in the relation of a transformation R. Bianchi[2] has shown that when S is applicable to a surface of revolution, it is possible to choose spheres so that for every deformation of S the two sheets of the envelopes of the spheres shall be in the relation of a transformation R, and this is the only case in which S can be deformed continuously with transformations R preserved. The only other possibility is that in which it is possible to deform the surface of centers of a transformation R in one way so that the sheets of the new envelope shall be in the relation of a transformation R. It is the purpose of this paper to determine this class of transformations R.

Let x, y, z denote the cartesian coordinates of Σ, expressed in terms of two parameters u and v, such that the curves u = const., v = const. are the lines of curvature on Σ and let the linear element of Σ be written $ds^2 = Edu^2 + Gdv^2$. If X, Y, Z are the direction-cosines of the normal to Σ and ρ_1, ρ_2 its principal radii of normal curvature, we have the equations of Rodriques

$$\frac{\partial x}{\partial u} + \rho_1 \frac{\partial X}{\partial u} = 0, \qquad \frac{\partial x}{\partial v} + \rho_2 \frac{\partial X}{\partial v} = 0, \tag{1}$$

and similar equations[3] in y and z.

Darboux[4] has shown that the most general transformation R of Σ is given by taking for S the surface whose coordinates x_0, y_0, z_0 are given by

$$x_0 = x - \frac{\lambda}{\mu} X, \qquad y_0 = y - \frac{\lambda}{\mu} Y, \qquad z_0 = z - \frac{\lambda}{\mu} Z,$$

where λ and μ satisfy the equations

$$\frac{\partial \lambda}{\partial u} + \rho_1 \frac{\partial \mu}{\partial u} = 0, \qquad \frac{\partial \lambda}{\partial v} + \rho_2 \frac{\partial \mu}{\partial v} = 0, \tag{2}$$

and by taking λ/μ for the radius of the corresponding sphere.

If x_1, y_1, z_1, denote the coordinates of Σ_1, and X_1, Y_1, Z_1; X_2, Y_2, Z_2, the direction-cosines of the tangents to the curves v = const., u = const. respectively on Σ, we have relations of the form

$$x_1 = x - \frac{1}{\sigma m}(\alpha X_1 + \beta X_2 + \mu X),$$

where m is a constant, and α, β, σ are functions which are in the quadratic relation

$$\alpha^2 + \beta^2 + \mu^2 = 2m\lambda\sigma,$$

and satisfy the equations

$$\begin{aligned} &\frac{\partial \lambda}{\partial u} = \sqrt{E}\,\alpha, \qquad &&\frac{\partial \lambda}{\partial v} = \sqrt{G}\,\beta, \\ &\frac{\partial \mu}{\partial u} = -\frac{\sqrt{E}}{\rho_1}\alpha, \qquad &&\frac{\partial \mu}{\partial v} = -\frac{\sqrt{G}}{\rho_2}\beta, \\ &\frac{\partial \alpha}{\partial u} = -\frac{1}{\sqrt{G}}\frac{\partial \sqrt{E}}{\partial v}\beta + \mu\frac{\sqrt{E}}{\rho_1} + 2m\sqrt{\lambda\sigma}\cosh\omega, && \end{aligned} \tag{3}$$

$$\frac{\partial\alpha}{\partial v} = \frac{\beta}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}, \qquad \frac{\partial\beta}{\partial u} = \frac{\alpha}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v},$$

$$\frac{\partial\beta}{\partial v} = -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\,\alpha + \mu\frac{\sqrt{G}}{\rho_2} + 2m\sqrt{\lambda\sigma}\sinh\omega, \tag{3}$$

$$\frac{\partial\log\sigma}{\partial u} = \left(2\sqrt{\frac{\lambda}{\sigma}}\cosh\omega - \sqrt{E}\right)\frac{\alpha}{\lambda}, \quad \frac{\partial\log\sigma}{\partial v} = \left(2\sqrt{\frac{\lambda}{\sigma}}\sinh\omega - \sqrt{G}\right)\frac{\beta}{\lambda},$$

$$\frac{\partial\omega}{\partial u} = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} - \frac{\alpha}{\sqrt{\lambda\sigma}}\sinh\omega, \quad \frac{\partial\omega}{\partial v} = \frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v} - \frac{\beta}{\sqrt{\lambda\sigma}}\cosh\omega.$$

This system of equations is completely integrable provided that

$$\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right).$$

This condition characterizes surfaces with the same spherical representation of their lines of curvature as isothermic surfaces. Then we establish the fundamental theorem:

In order that the surface of centers of a transformation R admits a deformation into the surface of centers of a transformation R, it is necessary that the sheets of the envelope of spheres have the same spherical representation of their lines of curvature as isothermic surfaces, and every surface of this sort admits such transformations.

From (1) and (2) it follows that μ is a solution of the differential equation satisfied by X, Y, Z, and consequently when a transformation R of a surface Σ is known, a transformation R of a surface $\bar{\Sigma}$ with the same spherical representation of its lines of curvature as Σ is determined by μ, the corresponding function λ being given by equations analogous to (2) and the other functions follow from equations of the form (3). We say that the new transformation R is obtained from the given one by a transformation of Combescure. Hence if we determine the solutions of our problem for which Σ is an isothermic surface, the others may be obtained by transformations of Combescure.

When Σ is isothermic, we may take

$$\sqrt{E} = \sqrt{G} = e^{\varphi},$$

in which case the integral of the last two of equations (3) is

$$e^{\omega} = \frac{\sqrt{\lambda\sigma}\,e^{\varphi}}{\lambda + c} \tag{4}$$

where c denotes a constant of integration.

When we take $c = 0$ in (4), and substitute this value of ω in (3), the latter reduced to the equations of transformations D_m of the isothermic surface Σ into a new isothermic surface Σ_1.[5] In this case the linear element of the sheet Σ' of the new envelope arising from the deformation of the surface of centers S of the transformation D_m is

$$ds'^2 = \left(e^{\varphi} - \frac{\lambda}{\sigma} e^{-\varphi}\right)^2 du^2 + \left(e^{\varphi} + \frac{\lambda}{\sigma} e^{-\varphi}\right)^2 dv^2,$$

and the linear element of its spherical representation is

$$d\sigma'^2 = \left(\frac{e^{\varphi}}{\rho_1} + \frac{\mu}{\sigma} e^{-\varphi}\right)^2 du^2 + \left(\frac{e^{\varphi}}{\rho_2} - \frac{\mu}{\sigma} e^{-\varphi}\right)^2 dv^2. \qquad (5)$$

The other sheet of the envelope, namely Σ'_1, is merely a point, say O. Conversely as we have shown elsewhere[6] also, a D_m is the only transformation R whose surface of centers can be deformed so that one of the sheets of the new envelope is a point.

From our fundamental theorem it follows that (5) is the spherical representation of the lines of curvature of two isothermic surfaces $\bar{\Sigma}$ and $\bar{\Sigma}'$, themselves Christoffel transforms of one another. Their respective linear elements are

$$d\bar{s}^2 = \sigma^2 e^{2\varphi}(du^2 + dv^2), \qquad d\bar{s}'^2 = \frac{e^{-2\varphi}}{\sigma^2}(du^2 + dv^2).$$

When we apply to $\bar{\Sigma}$ and $\bar{\Sigma}'$ transformations D_m determined by the function μ which is involved in the transformation from Σ' into the point O, we get two new isothermic surfaces $\bar{\Sigma}_1$ and $\bar{\Sigma}'_1$, whose linear elements are respectively

$$d\bar{s}_1^2 = \lambda^2 e^{-2\varphi}(du^2 + dv^2), \qquad d\bar{s}_1'^2 = \frac{e^{2\varphi}}{\lambda^2}(du^2 + dv^2).$$

The surface $\bar{\Sigma}_1'$ is the one which Bianchi[7] defined in a purely intrinsic manner and called the transform of $\bar{\Sigma}$ by the transformation T_m determined by the D_m of Σ into Σ_1.

Let S_0 be the surface of centers of the spheres enveloped by Σ' and O. Corresponding normals to Σ', $\bar{\Sigma}$, $\bar{\Sigma}'$ are parallel, and in like manner the normals to $\bar{\Sigma}_1$ and $\bar{\Sigma}'_1$ are parallel to the lines joining O to the points on S_0. Hence the permanent conjugate system on S_0, that is the system corresponding to the lines of curvature on Σ, project upon the unit sphere with center at O into the orthogonal curves representing the lines of curvature on $\bar{\Sigma}'_1$.

The surfaces $\bar{\Sigma}'$ and $\bar{\Sigma}'_1$ are in the relation of a transformation D_m. Let $\bar{S}$ be the surface of centers of the spheres of this transformation and $\bar{S}_0$ the deform of $\bar{S}$. Applying the above method to these surfaces, we find that the lines joining O to the points of $\bar{S}_0$ are parallel to the normals to Σ. Moreover, the surfaces $\bar{S}$ and S_0 correspond with parallelism of tangent planes; likewise $\bar{S}_0$ and S.

When $c \neq 0$ in (4), the surface Σ'_1 is an isothermic surface in the relation of a transformation T_m with Σ.

[1] Bianchi, *Geometria Differenziale*, **2**, 88.

[2] l. c., p. 117.

[3] Eisenhart, *Differential Geometry*, 122.

[4] *Théorie Générale des Surfaces*, **2**, 383.

[5] Darboux, *Ann. École Normale Supérieure*, ser. 3, **16** (1899).

[6] *Roma, Rend. Acc. Lincei*, ser. 5, **24**, 161 (1915).

[7] *Ann. Mat., Milano*, ser. 3, **12**, 19 (1906).

GEOGRAPHIC HISTORY OF THE SAN JUAN MOUNTAINS SINCE THE CLOSE OF THE MESOZOIC ERA

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The San Juan Mountains are located in the southwestern corner of Colorado. From their lofty summits one may look far off into Utah, Arizona, and New Mexico. The highest peak in the range, Uncompahgre, rises to an elevation of 14,300 feet above sea level. There are many other peaks whose summits are above 14,000 feet in elevation, and a large portion of the mountain area is more than 12,000 feet above sea level.

The core of the San Juan Mountains is composed of a complex series of pre-Cambrian gneisses, schists, quartzites, and granites, all of which have been more or less affected by intrusive bodies of igneous rocks. After this pre-Cambrian complex was so eroded that the relief in the region was low, the entire district was submerged and a great series of Paleozoic and Mesozoic strata was deposited over the once mountainous area.

At the close of the Mesozoic era and the opening of the Cenozoic era there were mountain-making movements which affected the entire Rocky Mountain province of North America, and the great dome which was then formed in the San Juan area was at once subjected to vigorous erosion. As the mountain mass rose erosion began, and as the great dome was more and more deeply dissected a mountain topog-